Exercise 3

Use residues to derive the integration formulas in Exercises 1 through 6.

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{dx}{x^4+1} = \int_{-\infty}^\infty \frac{dx}{2(x^4+1)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{1}{2(z^4+1)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$2(z^4+1) = 0$$
$$z^4+1 = 0$$
$$z = \sqrt[4]{1}\exp\left[i\left(\frac{\pi+2k\pi}{4}\right)\right], \quad k = 0, 1, 2, 3 \quad \rightarrow \quad \begin{cases} z_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\ z_2 = e^{i3\pi/4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\ z_3 = e^{i5\pi/4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \\ z_4 = e^{i7\pi/4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \end{cases}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = z_1$ and $z = z_2$.

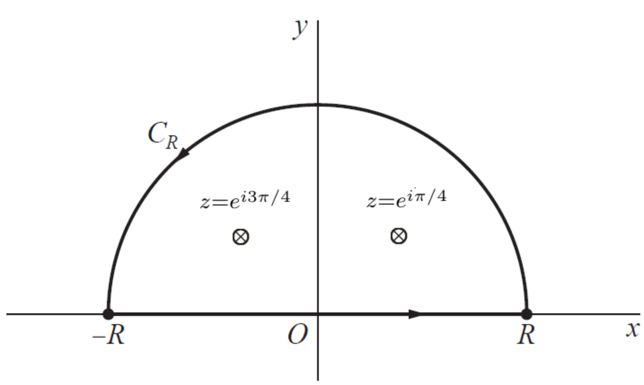

Figure 1: This is Fig. 99 with the singularities at $z = z_1$ and $z = z_2$ marked.

According to Cauchy's residue theorem, the integral of $1/[2(z^4+1)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{2(z^4+1)} = 2\pi i \left[\operatorname*{Res}_{z=z_1} \frac{1}{2(z^4+1)} + \operatorname*{Res}_{z=z_2} \frac{1}{2(z^4+1)}\right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{dz}{2(z^4+1)} + \int_{C_R} \frac{dz}{2(z^4+1)} = 2\pi i \left[\operatorname*{Res}_{z=z_1} \frac{1}{2(z^4+1)} + \operatorname*{Res}_{z=z_2} \frac{1}{2(z^4+1)}\right]$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, & r = -R \quad &\to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, & \theta = 0 \quad &\to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{dr}{2(r^4+1)} + \int_{C_R} \frac{dz}{2(z^4+1)} = 2\pi i \left[\operatorname*{Res}_{z=z_1} \frac{1}{2(z^4+1)} + \operatorname*{Res}_{z=z_2} \frac{1}{2(z^4+1)}\right].$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{dr}{2(r^4+1)} = 2\pi i \left[\operatorname*{Res}_{z=z_1} \frac{1}{2(z^4+1)} + \operatorname*{Res}_{z=z_2} \frac{1}{2(z^4+1)}\right]$$

The denominator can be written as $2(z^4+1) = 2(z-z_1)(z-z_2)(z-z_3)(z-z_4)$. From this we see that the multiplicities of the $z - z_1$ and $z - z_2$ factors are both 1. The residues at $z = z_1$ and $z = z_2$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=z_1} \frac{1}{2(z^4+1)} &= \phi_1(z_1) \\ \operatorname*{Res}_{z=z_2} \frac{1}{2(z^4+1)} &= \phi_2(z_2), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z - z_1$ and $z - z_2$ factors, respectively.

$$\phi_1(z) = \frac{1}{2(z-z_2)(z-z_3)(z-z_4)} \quad \Rightarrow \quad \phi_1(z_1) = \frac{1}{2(\sqrt{2})[\sqrt{2}(1+i)](i\sqrt{2})} = -\frac{1}{8\sqrt{2}}(1+i)$$

$$\phi_2(z) = \frac{1}{2(z-z_1)(z-z_3)(z-z_4)} \quad \Rightarrow \quad \phi_2(z_2) = \frac{1}{2(-\sqrt{2})(i\sqrt{2})[\sqrt{2}(-1+i)]} = \frac{1}{8\sqrt{2}}(1-i)$$

So then

$$\begin{aligned} \operatorname*{Res}_{z=z_1} \frac{1}{2(z^4+1)} &= -\frac{1}{8\sqrt{2}}(1+i) \\ \operatorname*{Res}_{z=z_2} \frac{1}{2(z^4+1)} &= \frac{1}{8\sqrt{2}}(1-i) \end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dr}{2(r^4+1)} &= 2\pi i \left[-\frac{1}{8\sqrt{2}}(1+i) + \frac{1}{8\sqrt{2}}(1-i) \right] \\
&= 2\pi i \left(\frac{-i}{4\sqrt{2}} \right) \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned} \int_{C_R} \frac{dz}{2(z^4+1)} &= \int_0^\pi \frac{Rie^{i\theta}\, d\theta}{2[(Re^{i\theta})^4+1]} \\ &= \int_0^\pi \frac{Rie^{i\theta}}{R^4 e^{i4\theta}+1} \frac{d\theta}{2} \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{dz}{2(z^4+1)} \right| &= \left| \int_0^\pi \frac{Rie^{i\theta}}{R^4 e^{i4\theta}+1} \frac{d\theta}{2} \right| \\ &\le \int_0^\pi \left| \frac{Rie^{i\theta}}{R^4 e^{i4\theta}+1} \right| \frac{d\theta}{2} \\ &= \int_0^\pi \frac{\left| Rie^{i\theta} \right|}{|R^4 e^{i4\theta}+1|} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{|R^4 e^{i4\theta}+1|} \frac{d\theta}{2} \\ &\le \int_0^\pi \frac{R}{|R^4 e^{i4\theta}| - |1|} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{R^4-1} \frac{d\theta}{2} \\ &= \frac{\pi}{2} \frac{R}{R^4-1} \end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{C_R} \frac{dz}{2(z^4+1)} \right| &\le \lim_{R\to\infty} \frac{\pi}{2} \frac{R}{R^4-1} \\ &= \lim_{R\to\infty} \frac{\pi}{2R^3} \frac{1}{1-\frac{1}{R^4}} \end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty} \left| \int_{C_R} \frac{dz}{2(z^4+1)} \right| \le 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty} \left| \int_{C_R} \frac{dz}{2(z^4+1)} \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{2(z^4+1)}\, dz = 0.$$